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CONTROLLED K-FRAMES IN 2-HILBERT SPACES

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Abstract: In this paper, we introduce a new generalization of controlled K-frames to the context of 2-Hilbert spaces, thereby extending beyond classical Hilbert space theory. We develop foundational results by examining the operator-theoretic properties of controlled K-frames in this setting, establishing equivalent conditions that characterize them, and exploring their stability under suitable transformations. This builds directly on prior work introducing controlled K-frames in Hilbert C^*-modules, where the concept was first defined, equivalent conditions were established, relationships between K-frames and controlled K-frames were revealed, and invariance and perturbation properties were analyzed. Our work elevates these ideas by adapting them to the richer structure of 2-Hilbert spaces-a framework extending Hilbert spaces through inner products valued in C^*-algebras.

Keywords and Phrases: Frame, Controlled K-frame, Controlled K-frame operator, Controlled K- Bessel sequence.

2020 Mathematics Subject Classification: 42C15, 46C50.

1. Introduction

Frames represent generalizations of the fundamental concept of bases within the study of vector spaces, initially formulated by Duffin and Schaeffer [5] during the 20th century in the context of non-harmonic Fourier series. This pioneering work laid the foundation for further exploration. Daubechies, Grossmann, and Meyer [3] continued to develop the idea, shedding light on frames. Today, frames play a crucial role not only in pure mathematics but also in applied mathematics, finding applications in signal and image processing [6], harmonic analysis [9], wireless communications [10], and various other fields.

Diminnie, Gahler, and White [4] introduced the concept of 2-inner product, paving the way for subsequent discussions. Building on this, Misiak [11] extended the notion of 2-inner product spaces to cases where $n \geq 2$.

Numerous generalizations have been proposed by researchers, such as K-frames [8], K-d-frames [15, 16], approximative frames [17], approximative K-frames [13, 14], controlled frames [1], and more.

The concept of controlled frames was initially introduced by Balazs, Antoine, and Grybos [1] with the aim of enhancing the numerical efficiency of iterative algorithms used for inverting the frame operator. Subsequently, the idea of the controlled K-frame was developed by Nouri, Rahimi, and Najafzadeh [12]. Building on the foundation laid by the literature on frames, we introduce the novel concept of controlled K-frames within the context of two Hilbert spaces.

2. Preliminaries

Throughout this paper, $\mathcal{H}$ denotes separable Hilbert space, $\mathcal{B}(\mathcal{H})$ a collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ and $\ell^2(\mathbb{N})$ denotes the square summable scalar valued sequences with index set of natural numbers $\mathbb{N}$. K^* is an adjoint of K. We denote $GL(\mathcal{H}_1, \mathcal{H}_2)$ as the set of all bounded linear operators with a bounded inverse, $GL(\mathcal{H})$ the set of all bounded linear operators with a bounded inverse from $\mathcal{H}$ to $\mathcal{H}$ and $GL^+(\mathcal{H})$ be the set of positive operators in $GL(\mathcal{H})$. $\mathbb{I}$ is the countable index set.

Definition 1. [2] *Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{x_i\}_{i\in\mathbb{I}}$ is called a frame for $\mathcal{H}$, if there exist constants $b, B > 0$ such that*

$$b||x||^2 \leq \sum_{i\in\mathbb{I}} |\langle x, x_i\rangle|^2 \leq B||x||^2, \quad \forall\, x \in \mathcal{H},$$

here, constants b and B are called lower and upper frame bounds respectively. If $\{x_i\}_{i\in\mathbb{I}}$ satisfies

$$\sum_{i\in\mathbb{I}} |\langle x, x_i\rangle|^2 \leq B||x||^2, \quad \forall\, x \in \mathcal{H},$$

then $\{x_i\}_{i\in\mathbb{I}}$ is called a Bessel sequence with bound B.

Definition 2. [2] *Let sequence $\{x_i\}_{i\in\mathbb{I}}$ be a frame for $\mathcal{H}$. Then the bounded linear operator $T : \mathcal{H} \to \ell^2(\mathbb{N})$, defined by $Tx = \{\langle x, x_i\rangle\}_{i\in\mathbb{I}}$ is called analysis operator and its adjoint operator $T^* : \ell^2(\mathbb{N}) \to \mathcal{H}$, defined by $T^*(\{c_i\})_{i\in\mathbb{I}} = \sum_{i\in\mathbb{I}} c_i x_i$ is called synthesis or pre-frame operator. Then operator $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ defined by $\mathcal{S}x = TT^*x = \sum_{i\in\mathbb{I}}\langle x, x_i\rangle x_i$ is called the frame operator.*

Definition 3. [4] *Let $\mathcal{X}$ be a linear space of dimension greater than 1 over the field $\mathbb{K}$($\mathbb{K}$ is the real or complex numbers field). Let us consider $\langle ., .|.\rangle$ is a $\mathbb{K}$-valued function defined on $\mathcal{X}\times\mathcal{X}\times\mathcal{X}$ satisfying the following conditions:-*
(i). $\langle x, x|z\rangle \geq 0$ *and* $\langle x, x|z\rangle = 0$ *iff x and z are linearly dependent,*
(ii). $\langle x, x|z\rangle = \underline{\langle z, z|x\rangle}$,
(iii).$\langle y, x|z\rangle = \overline{\langle x, y|x\rangle}$,
(iv). $\langle \alpha x, x|z\rangle = \alpha\langle x, y|z\rangle, \forall\alpha \in \mathbb{K}$,
(v). $\langle x_1 + x_2, y|z\rangle = \langle x_1, y|z\rangle + \langle x_2, y|z\rangle$.

$\langle ., .|.\rangle$ is called a 2-inner product on $\mathcal{X}$ and $(\mathcal{X}, \langle ., .|.\rangle)$ is called 2-inner product space(or 2-pre Hilbert space). Some basic properties of 2-inner product $\langle ., .|.\rangle$ can be immediately obtained as follows ().
(i). $\langle 0, y|z\rangle = \langle x, 0|z\rangle = \langle x, y|0\rangle = 0$,
(ii). $\langle x, \alpha y|z\rangle = \bar{\alpha}\langle x, y|z\rangle$,
(iii). $\langle x, y|\alpha z\rangle = |\alpha|^2\langle x, y|z\rangle, \forall x, y, z \in \mathcal{X}$ *and* $\alpha \in \mathbb{K}$.
Using these properties, we can prove the Cauchy Schwarz inequality.

$$|\langle x, y|z\rangle|^2 \leq \langle x, x|z\rangle\langle y, y|z\rangle.$$

Example 1. If $(\mathcal{X}, \langle ., .\rangle)$ is an inner product space, then the standard 2-inner product $\langle ., .|.\rangle$ is defined on $\mathcal{X}$ by

$$\langle x, y|z\rangle = \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{vmatrix} = \langle x, y\rangle\langle z, z\rangle - \langle x, z\rangle\langle z, y\rangle, \forall x, y, z \in \mathcal{X}.$$

In any given 2-inner product space $(\mathcal{X}, \langle ., .|.\rangle)$, we can define a function $\|., .\|$ defined on $\mathcal{X}\times\mathcal{X}$ by

$$\|x, z\| = \langle x, x|z\rangle^{1/2}, \forall x, y, z \in \mathcal{X}.$$

Definition 4. [7] *Any function $\|., .\|$ defined on $\mathcal{X}\times\mathcal{X}$ and satisfying the following conditions is called 2-norm on $\mathcal{X}$ and $(\mathcal{X}, \|., .\|)$ is called 2-normed space.*
(i). $\|x, z\| \geq 0$ *and* $\|x, z\| = 0$ *if and only if x and z are linearly dependent,*
(ii). $\|x, z\| = \|z, x\|$,

(iii). $\|\alpha x, z\| = |\alpha|\|x, z\|, \forall \alpha \in \mathbb{K}$, *(iv).* $\|x_1 + x_2, z\| = \|x_1 + z\| + \|x_2 + z\|$

Definition 5. [12] *Let* $C \in GL^+(\mathcal{H})(C > 0), K \in \mathcal{B}(\mathcal{H})$ *and* $CK = KC$. *A family* $\{x_i\}_{i\in\mathbb{I}} \in \mathcal{H}$ *is a controlled* K*-frames if there exist* $b > 0, B < \infty$ *such that*

$$b\|C^{1/2}K^*x\|^2 \leq \sum_{i\in\mathbb{I}} \langle x, x_i\rangle\langle Cx_i, x\rangle \leq B\|x\|^2, \quad \forall x \in \mathcal{H}, \tag{1}$$

where b *and* B *are lower and upper bounds of controlled* K*-frames for* $\mathcal{H}$.

Remark 1. *(i). If* $C = I, \{x_i\}_{i\in\mathbb{I}}$ *is called* K*-frame for* $\mathcal{H}$ *with bounds* b *and* B. *(ii).]If* $\sum_{i\in\mathbb{I}} \langle x, x_i\rangle\langle Cx_i, x\rangle \leq B\|x\|^2$, *then* $\{x_i\}_{i\in\mathbb{I}}$ *is called controlled* K*-Bessel sequence for* $\mathcal{H}$ *with bound* B.

3. Controlled K-Frame in 2-Hilbert Space

In this section, we introduce the notion of controlled K-frame in 2 Hilbert space. For the rest of the part, we assume $\langle \mathcal{X}, .|.\rangle$ is 2- Hilbert space and L_ρ the subspace generated with ρ for a fix element ρ in $\mathcal{X}$. M_ρ be the algebraic complement of L_ρ in $\mathcal{X}$. So $L_\rho \oplus M_\rho = \mathcal{X}$.
Define $\langle x, y\rangle_\rho$ on $\mathcal{X}$ as following:

$$\langle x, y\rangle_\rho = \langle x, y|\rho\rangle.$$

The semi-inner product induces an inner product on the quotient space $\mathcal{X}|L_\rho$ as

$$\langle x + L_\rho, y + L_\rho\rangle_\rho = \langle x, y\rangle_\rho = \langle x, y|\rho\rangle, \forall \quad x, y \in \mathcal{X}.$$

By introducing $\mathcal{X}|L_\rho$ with M_ρ is an obvious way, we obtain an inner product on M_ρ. Define $\|x\|_\rho = \sqrt{\langle x, x\rangle_\rho} = \sqrt{\langle x, x|\rho}, x \in M_\rho$. Then $(M_\rho, \|.\|_\rho)$ is a norm space. Let $\mathcal{X}_\rho$ be the completion of the inner product space M_ρ.

Definition 6. *Let* $C \in GL^+(\mathcal{X}_\rho)(C > 0), K \in \mathcal{B}(\mathcal{X}_\rho)$ *and* $CK = KC$. *Let* $(\mathcal{X}, \langle ., .|.\rangle)$ *be a Hilbert space and* $\rho \in \mathcal{X}$. *A family* $\{x_i\}_{i\in\mathbb{I}} \in \mathcal{X}$ *is a controlled* K*-frames associated to* ρ *if there exist* $b > 0, B < \infty$ *such that*

$$b\|C^{1/2}K^*x, \rho\|^2 \leq \sum_{i\in\mathbb{I}} \langle x, x_i|\rho\rangle\langle Cx_i, x|\rho\rangle \leq B\|x, \rho\|^2, \quad \forall x \in \mathcal{X}_\rho, \tag{2}$$

where b *and* B *are lower and upper bounds of controlled* K*-frames associated to* ρ *for* $\mathcal{X}$.

Remark 2. *If* $C = I, \{x_i\}_{i\in\mathbb{I}}$ *is called* K*-frame associated to* ρ *for* $\mathcal{X}$ *with bounds* b *and* B.

Remark 3. *If* $\sum_{i\in\mathbb{I}}\langle x, x_i|\rho\rangle\langle Cx_i, x|\rho\rangle \leq B\|x,\rho\|^2$, *then* $\{x_i\}_{i\in\mathbb{I}}$ *is called controlled* K*-Bessel sequence associated to* ρ *for* $\mathcal{X}$ *with bound* B.

Remark 4. *If* $b\|C^{1/2}K^*x,\rho\|^2 = \sum_{i\in\mathbb{I}}\langle x, x_i|\rho\rangle\langle Cx_i, x|\rho\rangle$ *in equation (2) then* $\{x_i\}_{i\in\mathbb{I}}$ *is called controlled* K*-tight frame associated to* ρ *for* $\mathcal{X}$ *with bound* b.

Remark 5. *Let* $\{x_i\}_{i\in\mathbb{I}}$ *is called controlled* K*-tight frame associated to* ρ *for* $\mathcal{X}$ *with bound* b. *Then* $\forall x \in \mathcal{X}_\rho$, *we have*

$$
\begin{aligned}
& \sum_{i\in\mathbb{I}}\langle x, x_i|\rho\rangle\langle Cx_i, x|\rho\rangle = b\|K^*C^{1/2}x,\rho\|^2 \\
\implies\quad & \frac{1}{b}\sum_{i\in\mathbb{I}}\langle x, x_i|\rho\rangle\langle Cx_i, x|\rho\rangle = \|K^*C^{1/2}x,\rho\|^2 \\
\implies\quad & b^{-1}\sum_{i\in\mathbb{I}}\langle x, x_i|\rho\rangle\langle Cx_i, x|\rho\rangle = \|K^*C^{1/2}x,\rho\|^2 \\
\implies\quad & \sum_{i\in\mathbb{I}}\langle x, b^{-1/2}x_i|\rho\rangle\langle Cb^{-1/2}x_i, x|\rho\rangle = \|K^*C^{1/2}x,\rho\|^2.
\end{aligned}
$$

$\implies \{b^{-1/2}x_i\}_{i\in\mathbb{I}}$ *is a controlled* K*-Parseval frame associated to* ρ *for* $\mathcal{X}$.

4. Controlled K-Frame Operators for 2-Hilbert Space

Let $\{x_i\}_{i\in\mathbb{I}}$ be a controlled K-Bessel sequence associated to ρ for $\mathcal{X}$. The operator $T_{\mathcal{X}_\rho} : \mathcal{X}_\rho \to \ell^2(\mathbb{N})$ defined by,

$$T_{\mathcal{X}_\rho}x = \{\langle x, x_i|\rho\rangle\}_{i\in\mathbb{I}}, x \in \mathcal{X}_\rho.$$

$T_{\mathcal{X}_\rho}$ is called analysis operator.
Synthesis or pre-controlled K-frame operator $T^*_{\mathcal{X}_\rho} : \ell^2(\mathbb{N}) \to \mathcal{X}_\rho$

$$T^*_{\mathcal{X}_\rho}(\{a_i\})_{i\in\mathbb{I}} = \sum_{i\in\mathbb{I}} a_i Cx_i.$$

The controlled K-frame operator $\mathcal{S}_{\mathcal{X}_\rho} : \mathcal{X}_\rho \to \mathcal{X}_\rho$

$$\mathcal{S}_{\mathcal{X}_\rho}x = T^*_{\mathcal{X}_\rho}T_{\mathcal{X}_\rho}x = \sum_{i\in\mathbb{I}}\langle x, x_i|\rho\rangle Cx_i.$$

Proposition 1. *Let* $C \in GL^+(\mathcal{X}_\rho)(C > 0), K \in \mathcal{B}(\mathcal{X}_\rho)$ *and let* $\{x_i\}_{i\in\mathbb{I}}$ *be a controlled* K*-frame associated to* ρ *for* $\mathcal{X}$ *with bounds* b *and* B. *Then,* $\mathcal{S}_{\mathcal{X}_\rho}$ *is bounded.*

Proof. Let $\{x_i\}_{i\in\mathbb{I}}$ be a controlled K-frame associated to ρ for $\mathcal{X}$ with bounds b and B. So, $\forall x \in \mathcal{X}_\rho$

$$\begin{aligned} b\langle C^{1/2}K^*x, C^{1/2}K^*x|\rho\rangle \leq \sum_{i\in\mathbb{I}}\langle x, x_i|\rho\rangle\langle Cx_i, x|\rho\rangle \quad &\leq \quad B\langle x, x|\rho\rangle \\ bC\|K^*x, \rho\|^2 \leq \mathcal{S}_{\mathcal{X}_\rho} &\leq B\|x, \rho\|^2 \\ bCKK^*I \leq \mathcal{S}_{\mathcal{X}_\rho} &\leq BI. \end{aligned}$$

Corollary 1. *Let $C \in GL^+(\mathcal{X}_\rho)(C > 0), K \in \mathcal{B}(\mathcal{X}_\rho)$ and let $\{x_i\}_{i\in\mathbb{I}}$ be a controlled Bessel sequence associated to ρ in $\mathcal{X}$. Then $\{x_i\}_{i\in\mathbb{I}}$ is a controlled K-frame associated to ρ in $\mathcal{X}$ if and only if there exists $b > 0$ such that*

$$C\mathcal{S} \geq bCKK^*.$$

Proof. Let $\{x_i\}_{i\in\mathbb{I}}$ be a controlled K-frame associated to ρ in $\mathcal{X}$ with bounds b and B. Then, $\forall x \in \mathcal{X}_\rho$

$$\begin{aligned} & b\langle C^{1/2}K^*x, C^{1/2}K^*x|\rho\rangle \leq \sum_{i\in\mathbb{I}}\langle x, x_i|\rho\rangle\langle Cx_i, x|\rho\rangle \leq B\langle x, x|\rho\rangle \\ \iff \quad & b\langle CKK^*x, x|\rho\rangle \leq \langle \mathcal{S}_{\mathcal{X}_\rho}x, x|\rho\rangle \leq B\langle x, x|\rho\rangle \\ \iff \quad & b\langle CKK^*x, x|\rho\rangle \leq \langle C\mathcal{S}x, x|\rho\rangle \leq B\langle x, x|\rho\rangle \\ \iff \quad & bCKK^*I \leq C\mathcal{S} \leq BI \\ \iff \quad & bCKK^*I \leq C\mathcal{S}. \end{aligned}$$

Theorem 2. *Let $\{x_i\}_{i\in\mathbb{I}}$ be a controlled K-frame associated to ρ for $\mathcal{X}$ and $\mathcal{S}_{\mathcal{X}_\rho}$ be 2-controlled K-frame operator of $\{x_i\}_{i\in\mathbb{I}}$ where $KK^* = I$ If we denote the positive square root of $\mathcal{S}_{\mathcal{X}_\rho}^{-1}x_i$ by $\mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i$ then the $\{\mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i\}_{i\in\mathbb{I}}$ is a Parseval controlled K-frame associated to ρ.*

Proof. 2-Controlled K-frame operator

$$\mathcal{S}_{\mathcal{X}_\rho}x = \sum_{i\in\mathbb{I}}\langle x, x_i|\rho\rangle Cx_i$$

Putting $x = \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x$

$$\begin{aligned} \mathcal{S}_{\mathcal{X}_\rho}^{1/2}x &= \sum_{i\in\mathbb{I}}\langle \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x, x_i|\rho\rangle Cx_i \\ x &= \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}(\sum_{i\in\mathbb{I}}\langle x, \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i|\rho\rangle Cx_i) \\ &= \sum_{i\in\mathbb{I}}\langle x, \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i|\rho\rangle \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}Cx_i \end{aligned}$$

Now,

$$\begin{aligned}
\|C^{1/2}K^*x,\rho\|^2 &= \langle C^{1/2}K^*x, C^{1/2}K^*x|\rho\rangle \\
&= \langle C^{1/2}K^* \sum_{i\in\mathbb{I}} \langle x, \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i|\rho\rangle \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}Cx_i, C^{1/2}K^*x|\rho\rangle \\
&= \sum_{i\in\mathbb{I}} \langle x, \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i|\rho\rangle \langle C^{1/2}K^*\mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i, C^{1/2}K^*x|\rho\rangle \\
&= \sum_{i\in\mathbb{I}} \langle x, \mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i|\rho\rangle \langle C\mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i, x|\rho\rangle
\end{aligned}$$

Hence $\{\mathcal{S}_{\mathcal{X}_\rho}^{-1/2}x_i\}_{i\in\mathbb{I}}$ is a Parseval 2-controlled K-frame for $\mathcal{X}$.

Proposition 3. *Let $\{x_i\}_{i\in\mathbb{I}}$ be a controlled K-Bessel sequence associated to ρ for $\mathcal{X}$ with bound B. Let $M \in \mathcal{B}(\mathcal{X}_\rho)$ and $CM = MC$. Then, $\{Mx_i\}_{i\in\mathbb{I}}$ be a controlled K-Bessel sequence associated to ρ for $\mathcal{X}$ with bound $B\|M^*,\rho\|^2$.*

Proof. Let $\{x_i\}_{i\in\mathbb{I}}$ be a controlled K-Bessel sequence associated to ρ. Then, $\forall x \in \mathcal{X}_\rho$

$$\begin{aligned}
\sum_{i\in\mathbb{I}} \langle x, x_i|\rho\rangle\langle Cx_i, x|\rho\rangle &\leq B\langle x, x|\rho\rangle \\
\sum_{i\in\mathbb{I}} \langle x, Mx_i|\rho\rangle\langle CMx_i, x|\rho\rangle &= \sum_{i\in\mathbb{I}} \langle M^*x, x_i|\rho\rangle\langle MCx_i, x|\rho\rangle \\
&\leq B\langle M^*x, M^*x|\rho\rangle \\
&\leq B\|M^*,\rho\|^2\langle x, x|\rho\rangle.
\end{aligned}$$

Hence, $\{Mx_i\}_{i\in\mathbb{I}}$ be a controlled K-Bessel sequence associated to ρ for $\mathcal{X}$ with bound $B\|M^*,\rho\|^2$.

Theorem 4. *Let $K \in \mathcal{B}(\mathcal{X}_\rho)$ and let $\{x_i\}_{i\in\mathbb{I}}$ be a 2-controlled K-frame for $\mathcal{X}_\rho$. If $M \in \mathcal{B}(\mathcal{X}_\rho)$ has closed range with $R(K^*) \subset R(M), MK = KM$ and $MC = CM$, then $\{Mx_i\}_{i\in\mathbb{I}}$ is a 2-controlled K-frame for $R(M)$.*

Proof. We know that M has pseudo-inverse $M^\dagger$ since it has closed range such that

$$MM^\dagger = I_{R(M)}.$$

Now $I_{R(M)} = I^*_{R(M)} = (M^\dagger)^*M^*$. Since $R(K^*) \subset R(M)$ then for each $x \in R(M)$,

$$\begin{aligned}
K^*x &= (M^\dagger)^*M^*K^*x \\
\|K^*x,\rho\| &= \|(M^\dagger)^*M^*K^*x,\rho\| \leq \|(M^\dagger)^*\|\|M^*K^*x,\rho\|
\end{aligned}$$

It implies, $\|(M^\dagger)^*\|^{-1}\|K^*x,\rho\| \leq \|M^*K^*x,\rho\|$.
Now for each $x \in R(M)$,

$$\begin{aligned} b\|(M^\dagger)^*\|^{-2}\|K^*x,\rho\|^2 &= b\|M^*K^*x,\rho\|^2 \\ &\leq \sum_{i\in\mathbb{I}} \langle M^*x, x_i|\rho\rangle\langle Cx_i, M^*x|\rho\rangle \\ &= \sum_{i\in\mathbb{I}} \langle x, Mx_i|\rho\rangle\langle MCx_i, x|\rho\rangle \\ &= \sum_{i\in\mathbb{I}} \langle x, Mx_i|\rho\rangle\langle CMx_i, x|\rho\rangle \\ &\leq B\|M\|^2\|x,\rho\|^2. \end{aligned}$$

The following result shows that a controlled K-frame associated to ρ is a K-frame associated to ρ for $\mathcal{X}$.

Theorem 5. *Let $\{x_i\}_{i\in\mathbb{I}}$ be a controlled K-frame associated to ρ for $\mathcal{X}$ and $C \in GL^+(\mathcal{X}_\rho)$. Then, $\{x_i\}_{i\in\mathbb{I}}$ is a K-frame associated to ρ for $\mathcal{X}$.*

Proof. Let $\{x_i\}_{i\in\mathbb{I}}$ be a controlled K-frame associated to ρ for $\mathcal{X}$ with bounds A and B. Then, $\forall x \in \mathcal{X}_\rho$

$$\begin{aligned} b\|K^*x,\rho\|^2 &= b\|C^{-1/2}C^{1/2}K^*x,\rho\|^2 \\ &\leq b\|C^{1/2}\|^2\|C^{-1/2}K^*x,\rho\|^2 \\ &\leq \|C^{1/2}\|^2 \sum_{i\in\mathbb{I}} \langle x, x_i|\rho\rangle\langle C^0x_i, x|\rho\rangle \\ &= \|C^{1/2}\|^2 \sum_{i\in\mathbb{I}} |\langle x, x_i|\rho\rangle|^2. \end{aligned}$$

Hence,

$$b\|C^{1/2}\|^{-2}\|K^*x,\rho\|^2 \leq \sum_{i\in\mathbb{I}} |\langle x, x_i|\rho\rangle|^2.$$

For the upper bound,

$$\begin{aligned} \sum_{i\in\mathbb{I}} |\langle x, x_i|\rho\rangle|^2 &= \langle x, \mathcal{S}x|\rho\rangle = \langle x, C^{-1}C\mathcal{S}x|\rho\rangle \\ &= \langle (C^{-1}C\mathcal{S})^{1/2}x, (C^{-1}C\mathcal{S})^{1/2}x|\rho\rangle \\ &= \|(C^{-1}C\mathcal{S})^{1/2}x,\rho\|^2 \\ &\leq \|C^{-1/2}\|^2\|(C\mathcal{S})^{1/2}x,\rho\|^2 \\ &= \|C^{-1/2}\|^2\langle x, C\mathcal{S}x|\rho\rangle \\ &\leq \|C^{-1/2}\|^2B\|x,\rho\|^2. \end{aligned}$$

So, $\{x_i\}_{i\in\mathbb{I}}$ is a K-frame associated to ρ for $\mathcal{X}$. with bounds $b\|C^{1/2}\|^{-2}$ and $B\|C^{-1/2}\|^2$.

We discuss the following condition when a K-frame associated to ρ is a controlled K-frame associated to ρ for $\mathcal{X}$.

Theorem 6. *Let $C \in GL^+(\mathcal{X}_\rho)$ be a self-adjoint and $KC = CK$, if $\{x_i\}_{i\in\mathbb{I}}$ is a K-frame associated to ρ for $\mathcal{X}$ with bounds b and B, then $\{x_i\}_{i\in\mathbb{I}}$ is a controlled K-frame associated to ρ for $\mathcal{X}$ with bounds b and $B\|C\|$.*

Proof. Let $\{x_i\}_{i\in\mathbb{I}}$ be a K-frame associated to ρ for $\mathcal{X}$. Then, $\forall x \in \mathcal{X}_\rho$

$$\begin{aligned} b\|K^*x,\rho\|^2 &\leq \sum_{i\in\mathbb{I}} |\langle x, x_i|\rho\rangle|^2 \leq B\|x\|^2. \\ b\|C^{1/2}K^*x,\rho\|^2 = b\|K^*C^{1/2}x,\rho\|^2 &\leq \sum_{i\in\mathbb{I}} \langle C^{1/2}x, x_i|\rho\rangle\langle C^{1/2}x, x_i|\rho\rangle \\ &= \langle C^{1/2}x, \sum_{i\in\mathbb{I}} \langle x_i|\rho, C^{1/2}x\rangle x_i|\rho\rangle \\ &= \langle C^{1/2}x, C^{1/2}\mathcal{S}x|\rho\rangle = \langle x, C\mathcal{S}x|\rho\rangle. \end{aligned}$$

On the other hand, $\forall x \in \mathcal{X}_\rho$

$$\begin{aligned} |\langle x, C\mathcal{S}x|\rho\rangle|^2 &= |\langle C^*x, \mathcal{S}x|\rho\rangle|^2 = |\langle Cx, \mathcal{S}x|\rho\rangle|^2 \\ &\leq \|Cx,\rho\|^2\|\mathcal{S}x,\rho\|^2 \leq \|C\|^2\|x,\rho\|^2 B\|x,\rho\|^2. \end{aligned}$$

Thus,

$$b\|C^{1/2}K^*x,\rho\|^2 \leq \langle x, C\mathcal{S}x|\rho\rangle \leq B\|C\|\|x,\rho\|^2.$$

Hence, $\{x_i\}_{i\in\mathbb{I}}$ is a controlled K-frame associated to ρ for $\mathcal{X}$ with bounds b and $B\|C\|$.

5. Conclusion

This paper broadens the concept of controlled K-frames to the setting of 2-Hilbert spaces, providing a comprehensive analysis of their properties and associated operators. We reinforce the structural foundation of these frames in this extended framework. Furthermore, the conditions under which a K-frame in a 2-Hilbert space can also serve as a controlled K-frame are rigorously examined, contributing to a deeper understanding of the interplay between these two notions in the context of 2-Hilbert spaces. These advancements have potential applications in various fields, including cryptography and secure communication, seismic signal processing, and other domains requiring robust and flexible signal representations.

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